Міністерство освіти і науки України
Дніпровський національний університет
ім. Олеся Гончара

Traktynska V.M., Parfinovych N.V., Tkachenko M.E.

VECTOR ALGEBRA PRACTICE

Дніпро

2018

Уміщений теоретичний і практичний матеріал згідно з чинною програмою курсу “Вища математика”.

Для студентів І курсу технічних факультетів ДНУ.

Посібник рекомендований до друку Вченою радою механіко-математичного факультету від 12.09.2017 р., протокол №1.

Посібник

«VECTOR ALGEBRA PRACTICE»

Укладачі: канд. фіз.-мат. наук, доцент В.М. Трактинська
канд. фіз.-мат. наук, доцент Н.В. Парфінович
канд. фіз.-мат. наук, доцент М.Є. Ткаченко

Рецензенти: д-р фіз.-мат. наук, проф. С.Б. Вакарчук
канд. фіз.-мат. наук, проф. В.М. Турчин

У авторській редакції

Підписано до друку . Формат 60×84/16. Папір офсетний.
Друк цифровий. Ум. друк. арк. . Ум. фарбовідб. .
Обл.-вид. арк. . Тираж пр. Зам. № .

ПП «Ліра ЛТД», вул. Погребняка, 25 м. Дніпро, 49010
Свідоцтво про внесення до державного реєстру
Серія ДК №188 від19.09.2000 р. Фізична адреса: вул. Наукова, 5

FOREWORD

The section "Vector Algebra" is a part of the course "Higher Mathematics", which is studied by students of technical specialties of the University.

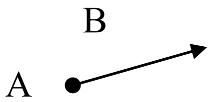
The “Vector Algebra Practice" provides theoretical information that covers topics such as Linear Operations on Vectors, Basis of Vector Space and Vector's Cartesian Components, Scalar,Vector and Scalar Triple Products of Vectors, examples of some typical problem solutions, as well as tasks for independent work.

This publication is aimed at helping English-speaking students to consolidate their knowledge and skills in solving vector algebra problems.

1. ALGEBRA OF VECTORS

1.1. Linear Operations on Vectors

A ***vector*** $\bar{a}$ is a mathematical object that has magnitude and direction. A directed line segment represents vector geometrically. A vector is associated with an ordered pair of points $\overline{AB}$. The point A is called ***the start point (or initial point)*** and B is ***the end point (or terminal point) of the vector*** $\overline{AB}$**.**



The length (magnitude) of $\bar{a}$ is its length and is normally denoted by $|\bar{a}| = |\overline{AB}|$. ***Zero vector or null vector*** is a vector that has zero magnitude and an arbitrary direction. It denoted by symbol $\bar{0}$. If $|\bar{a}| = 1$ then a vector $\bar{a}$ is called a ***unit vector***.

The vectors that lie on the same line or on parallel lines to the given one are called ***collinear vectors.*** Vectors that are parallel to the same plane, or lie in the same plane are called ***coplanar vectors.*** Otherwise, vectors are called **non-coplanar.**

The collinear vectors may have either same direction or opposite directions. When vectors have the same direction, they are called ***parallel vectors.*** The collinear vectors are called ***anti-parallel*** if they have the opposite directions.

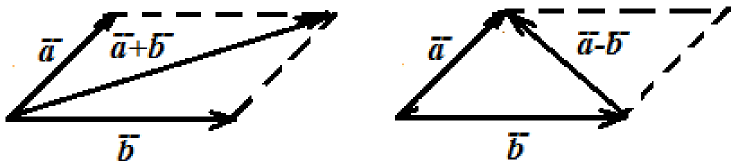
Unit vector that has same direction with $\bar{a}$ is called an ***orth of the vector*** $\bar{a}$.

Two vectors are called ***equal*** if they have equal magnitude and the same directions.

The vector $-\bar{a}$ is called ***the opposite*** of the vector $\bar{a}$, if it has the same magnitude as $\bar{a}$ but it has opposite direction.

Let the end point of the vector $\bar{a}$ be the same as the start point of the vector $\bar{b}$. We define ***sum*** $\bar{a} + \bar{b}$ of two vectors $\bar{a}$ and $\bar{b}$. Its start point matches with the start point of the vector $\bar{a}$, and the end point matches with the end point of the vector $\bar{b}$. Then ***the difference*** $\bar{a} - \bar{b}$ of vectors $\bar{a}$ and $\bar{b}$ is the vector $\bar{a} + (-\bar{b})$.

Parallelogram Method: Draw the vectors $\bar{a}$ and $\bar{b}$ so that their initial points coincide. Then draw lines to form a complete parallelogram. The vector diagonal from the initial point to the opposite vertex of the parallelogram is the vector $\bar{a}+\bar{b}$. Another diagonal is the vector $\bar{a}-\bar{b}$. It starts from the end point of the vector $\bar{b}$. Its terminal point is the end point of the vector $\bar{a}$.



Product $\alpha\bar{\boldsymbol{a}}$ ***of the vector*** $\bar{\boldsymbol{a}}$ ***by a nonnegative scalar*** α is a vector parallel to $\bar{a}$ with magnitude $\alpha|\bar{a}|$. For negative α, ***product*** $\alpha\bar{\boldsymbol{a}}$ is the vector anti-parallel to $\bar{a}$ with magnitude $|\alpha|\cdot|\bar{a}|$.

1.2. Basis of Vector Space and Vector's Cartesian Components

In 3D space any three ordered non-coplanar vectors $\bar{\boldsymbol{e}}_{\mathbf{1}}$, $\bar{\boldsymbol{e}}_{\mathbf{2}}$ and $\bar{\boldsymbol{e}}_{\mathbf{3}}$ form a ***basis***. Any vector from 3D space can be expressed as a linear combination:

$$\bar{a} = X\cdot\bar{e}_1 + Y\cdot\bar{e}_2 + Z\cdot\bar{e}_3. \tag{1}$$

The numbers X, Y, Z are called ***coordinates of the vector*** $\bar{\boldsymbol{a}}$ ***in this basis.*** The formula (1) is called ***decomposition of the vector*** $\bar{\boldsymbol{a}}$ ***on the vectors of the basis*** $\bar{\boldsymbol{e}}_{\mathbf{1}}, \bar{\boldsymbol{e}}_{\mathbf{2}}, \bar{\boldsymbol{e}}_{\mathbf{3}}$**.**

If vectors $\bar{\boldsymbol{e}}_{\mathbf{1}}, \bar{\boldsymbol{e}}_{\mathbf{2}}, \bar{\boldsymbol{e}}_{\mathbf{3}}$ are mutually orthogonal unit vectors then the basis $\bar{\boldsymbol{e}}_{\mathbf{1}}, \bar{\boldsymbol{e}}_{\mathbf{2}}, \bar{\boldsymbol{e}}_{\mathbf{3}}$ is called an ***orthonormal basis***. These vectors are orths of 3D space OXYZ and are denoted by $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}}, \bar{\boldsymbol{k}}$.

Let the points A_1 and B_1 be the orthogonal projections of the points A and B on the axis *p* respectively. **The projection** $\mathrm{proj}_p\bar{a}$ **of the vector** $\bar{\boldsymbol{a}}$ **on the axis** ***p*** is equal to $|\overline{A_1B_1}|$ $(-|\overline{A_1B_1}|)$, if $\overline{A_1B_1}$ and *p* are parallel (anti-parallel).

The projection of the vector $\bar{a}$ on the axis *p* is

$$\text{proj}_p \bar{a} = |\bar{a}| \cdot cos\varphi,$$

where φ is the angle between the vector $\bar{a}$ and the axis p.

The coordinates X, Y, Z of the vector $\bar{a}$ in the orthonormal basis are equal to the projections of the vector $\bar{a}$ on the axes OX, OY, OZ respectively. They are called ***Cartesian components of the vector*** $\boldsymbol{\bar{a}}$; this could be written as $\bar{a} = (X; Y; Z)$ or $\bar{a} = X \cdot \bar{\imath} + Y \cdot \bar{\jmath} + Z \cdot \bar{k}$.

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in OXYZ space. They are the initial point and the terminal point of the vector $\bar{a} = X \cdot \bar{\imath} + Y \cdot \bar{\jmath} + Z \cdot \bar{k}$ respectively. Then the Cartesian components of the vector $\bar{a}$ are computed by the formulas

$$X = x_2 - x_1, \qquad Y = y_2 - y_1, \quad Z = z_2 - z_1 .$$

The length (magnitude) of this vector is the number

$$|\bar{a}| = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The direction of the vector $\bar{a}$ is determined by angles α, β, γ between this vector and the coordinate axes OX, OY, OZ respectively. The cosines of these angles (the **direction cosines)** are calculated by the formulas

$$\cos\alpha = \frac{X}{|\bar{a}|} = \frac{X}{\sqrt{X^2 + Y^2 + Z^2}}, \qquad \cos\beta = \frac{Y}{|\bar{a}|} = \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}},$$

$$\cos\gamma = \frac{Z}{|\bar{a}|} = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}.$$

The direction cosines satisfy the following equality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 .$$

If $\bar{a} = (X_1; Y_1; Z_1)$, $\bar{b} = (X_2; Y_2; Z_2)$ then

$$\bar{a} + \bar{b} = (X_1 + X_2; Y_1 + Y_2; Z_1 + Z_2),$$

$$\bar{a} - \bar{b} = (X_1 - X_2; Y_1 - Y_2; Z_1 - Z_2),$$

$$\alpha \cdot \bar{a} = (\alpha X_1; \alpha Y_1; \alpha Z_1), \forall \alpha \in \mathbb{R}.$$

The vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if they are proportional, i.e.

$$\bar{b} = \lambda \bar{a} \qquad \text{or} \qquad \frac{X_2}{X_1} = \frac{Y_2}{Y_1} = \frac{Z_2}{Z_1}.$$

A number λ, such as $\overline{AM} = \lambda \cdot \overline{MB}$, is called ***ratio***, in which point M divides directed segment $\overline{AB}$. If $\lambda > 0$ then the point M belongs to the segment AB, if $\lambda < 0$ then the point M doesn't belong to it.

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and the point $M(x, y, z)$ divides $\overline{AB}$ in the ratio λ. Then

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \qquad y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \qquad z = \frac{z_1 + \lambda z_2}{1 + \lambda}.$$

1.3. Right-handed and Left-handed Triples

Suppose that $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors. Let O be the origin of the 3D-space, and define A, B, C by the conditions

$$\overline{OA} = \bar{a}, \qquad \overline{OB} = \bar{b}, \qquad \overline{OC} = \bar{c}.$$

Using your right hand put your thumb in the direction of $\bar{a}$, and your first [index] finger in the direction of $\bar{b}$. If C lies on the side of the plane through O, A, B indicated by your second [middle] finger, we call $\bar{a}, \bar{b}, \bar{c}$ a right-handed triple; otherwise it is a left-handed triple. For any triple $\bar{a}, \bar{b}, \bar{c}$ of vectors, precisely one of the following properties holds: it is coplanar, it is right-handed, or it is left-handed.

Examples. $\bar{\imath}, \bar{\jmath}, \bar{k}$ is a right-handed triple.

$\bar{\imath}, \bar{\jmath}, -\bar{k}$ and $\bar{k}, \bar{\jmath}, \bar{\imath}$ are both left-handed triples.

Note. The vectors $\bar{a}, \bar{b}, \bar{c}$ forming a right-handed or left-handed triple need not be mutually orthogonal, but they must be non-coplanar.

1.4. Scalar Product of Vectors

The scalar product (or dot product) of two vectors $\bar{a}$ and $\bar{b}$ is a number (scalar!), defined by the

$$(\bar{a}, \bar{b}) = |\bar{a}| \cdot |\bar{b}| \cdot \cos\varphi,$$

where φ is the angle between $\bar{a}$ and $\bar{b}$.

Another formula:

$$(\bar{a}, \bar{b}) = |\bar{a}| \cdot \mathrm{proj}_{\bar{\boldsymbol{a}}}\bar{b} = |\bar{b}| \cdot \mathrm{proj}_{\bar{b}}\bar{a}\,.$$

The properties of scalar product:

1. $(\bar{a}, \bar{b}) = (\bar{b}, \bar{a})$.
2. $(\bar{a}, \bar{b} + \bar{c}) = (\bar{a}, \bar{b}) + (\bar{a}, \bar{c})$.
3. $(\lambda\bar{a}, \bar{b}) = (\bar{a}, \lambda\bar{b}) = \lambda(\bar{a}, \bar{b})$.
4. $(\bar{a}, \bar{a}) = \bar{a}^{\mathbf{2}} = |\bar{a}|^2$.
5. $(\bar{a}, \bar{b}) = 0$ if and only if $\bar{a}$ is zero vector or $\bar{b}$ is zero vector or $\bar{a}$ and $\bar{b}$ are orthogonal vectors $(\bar{a} \perp \bar{b})$.

The scalar product of the basis orths:

$$\bar{\imath}^2 = \bar{\jmath}^2 = \bar{k}^2 = 1; \;\; (\bar{\imath}, \bar{\jmath}) = (\bar{\imath}, \bar{k}) = (\bar{\jmath}, \bar{k}) = 0.$$

Now let $\bar{a} = (X_1; Y_1; Z_1)$, $\bar{b} = (X_2; Y_2; Z_2)$ then

$$(\bar{a}, \bar{b}) = X_1X_2 + Y_1Y_2 + Z_1Z_2.$$

The condition of the orthogonality for non-zero vectors has the following form

$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0.$$

The angle between these vectors is calculated by the formula

$$\cos\varphi = \frac{(\bar{a},\bar{b})}{|\bar{a}|\cdot|\bar{b}|} = \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}\cdot\sqrt{X_2^2+Y_2^2+Z_2^2}}.$$

1.5. Vector Product of Vectors

Suppose that $\bar{a},\bar{b}$ are nonzero non-parallel vectors (of $\mathbb{R}^3$). Assume the angle between vectors $\bar{a}$ and $\bar{b}$ is equal to φ.

Then the **vector product** or **cross product** of the vectors $\bar{a}$ and $\bar{b}$ is the third vector $\bar{c} = [\bar{a},\bar{b}]$ defined as follows:

1) the magnitude of the vector $\bar{c}$ is equal to

$$|\bar{c}| = |\bar{a}|\cdot|\bar{b}|\cdot\sin\varphi\,;$$

(note that $\sin\varphi>0$ here, since $0<\varphi<\pi$);

2) vector $\bar{c}$ is orthogonal to both $\bar{a}$ and $\bar{b}$;

3) $\bar{a},\bar{b},\bar{c}$ is a right-handed triple.

If $\bar{a}$ or $\bar{b}$ is null vector or $\bar{a},\bar{b}$ are parallel, then the vector product is defined to be $\bar{0}$.

The properties of the vector product:

1. $[\bar{a},\bar{b}] = -[\bar{b},\bar{a}]$.
2. $[\bar{a},\bar{b}+\bar{c}] = [\bar{a},\bar{b}] + [\bar{a},\bar{c}]$.
3. $[\lambda\bar{a},\bar{b}] = [\bar{a},\lambda\bar{b}] = \lambda[\bar{a},\bar{b}]$.
4. $[\bar{a},\bar{b}] = \bar{0}$ if and only if $\bar{a}$ is zero vector or $\bar{b}$ is zero vector or $\bar{a}$ and $\bar{b}$ are collinear vectors.
5. The magnitude of the vector $[\bar{a},\bar{b}]$ is equal to the area of the parallelogram spanned by vectors $\bar{a}$ and $\bar{b}$.

The vector product of the basis orths:

$$[\bar{i},\bar{i}] = [\bar{j},\bar{j}] = [\bar{k},\bar{k}] = \bar{0},$$

$$[\bar{i},\bar{j}] = -[\bar{j},\bar{i}] = \bar{k}, \qquad [\bar{j},\bar{k}] = -[\bar{k},\bar{j}] = \bar{i}, \qquad [\bar{k},\bar{i}] = -\,[\bar{i},\bar{k}] = \bar{j}.$$

The vector product of the vectors $\bar{a} = (X_1; Y_1; Z_1)$ and $\bar{b} = (X_2; Y_2; Z_2)$ is defined according to the following formula

$$[\bar{a},\bar{b}] = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}.$$

1.6. Scalar Triple Product of Vectors

The **scalar triple product** of the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ is a number equal to $([\bar{a},\bar{b}],\bar{c}) = \bar{a}\bar{b}\bar{c}$.

The properties of scalar triple product:

1. $([\bar{a},\bar{b}],\bar{c}) = (\bar{a},[\bar{b},\bar{c}])$.
2. The scalar triple product is invariant under circular shift of its three operands:

$$\bar{a}\bar{b}\bar{c} = \bar{c}\bar{a}\bar{b} = \bar{b}\bar{c}\bar{a}\,.$$

3. While changing of the order of any two vectors of vector product only changes its sign:

$$\bar{b}\bar{a}\bar{c} = -\bar{a}\bar{b}\bar{c}.\ \bar{c}\bar{b}\bar{a} = -\bar{a}\bar{b}\bar{c}.\ \bar{a}\bar{c}\bar{b} = -\bar{a}\bar{b}\bar{c}.$$

4. The scalar triple product $\bar{a}\bar{b}\bar{c}$ is equal to zero if:

   – at least one of the vectors is equal to $\bar{0}$;

– two of the multiplied vectors are collinear;

– all the three vectors are coplanar.

5. $\bar{a}, \bar{b}, \bar{c}$ is a right-handed triple if and only if $\bar{a}\bar{b}\bar{c} > 0$;

   $\bar{a}, \bar{b}, \bar{c}$ is a left-handed triple if and only if $\bar{a}\bar{b}\bar{c} < 0$.

6. The absolute value of the scalar triple product is equal to the volume of the parallelepiped spanned by the vectors $\bar{a}, \bar{b}$ and $\bar{c}$:

$$V = |\bar{a}\bar{b}\bar{c}|.$$

7. The volume of the triangular pyramid determined by the vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is equal to

$$V = \frac{1}{6}|\bar{a}\bar{b}\bar{c}|.$$

If $\bar{a} = (X_1; Y_1; Z_1)$, $\bar{b} = (X_2; Y_2; Z_2), \bar{c} = (X_3; Y_3; Z_3)$ then

$$\bar{a}\bar{b}\bar{c} = \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}.$$

2. EXAMPLES

Example 1

Decompose the vector $\bar{x} = \bar{a} + \bar{b} + \bar{c}$ on the vectors $\bar{p} = \bar{a} + \bar{c}$, $\bar{q} = \bar{a} - \bar{b}$, $\bar{r} = \bar{a} + \bar{b} - 2\bar{c}$, where $\bar{a},\ \bar{b},\ \bar{c}$ are non-coplanar vectors.

Solution. Let's show that the vectors $\bar{p},\ \bar{q},\ \bar{r}$ form a basis (are non-coplanar). The decomposition of these vectors on vectors $\bar{a},\ \bar{b},\ \bar{c}$ gives their coordinates:

$$\bar{p} = (1; 0; 1), \qquad \bar{q} = (1; -1; 0), \qquad \bar{r} = (1; 1; -2).$$

Let's find

$$\bar{p}\bar{q}\bar{r} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = 4.$$

Since $\bar{p}\bar{q}\bar{r} \neq 0$, the vectors $\bar{p},\ \bar{q},\ \bar{r}$ are non-coplanar and form a basis.

Let's write a decomposition of the vector $\bar{x}$ on vectors $\bar{p},\ \bar{q},\ \bar{r}$:

$$\bar{x} = X \cdot \bar{p} + Y \cdot \bar{q} + Z \cdot \bar{r},$$

where X, Y, Z are unknown coordinates of the vector $\bar{x}$ on the basis $\bar{p},\ \bar{q},\ \bar{r}$.

Considering decompositions of $\bar{p},\ \bar{q},\ \bar{r}$ we have

$$\bar{x} = X \cdot (\bar{a} + \bar{c}\,) + Y \cdot \left(\bar{a} - \bar{b}\right) + Z \cdot \left(\bar{a} + \bar{b} - 2\bar{c}\right) =$$

$$= (X + Y + Z)\bar{a} + (-Y + Z)\bar{b} + (X - 2Z)\bar{c}.$$

Since $\bar{x} = \bar{a} + \bar{b} + \bar{c}$, then

$$\begin{cases} X + Y + Z = 1 \\ -Y + Z = 1 \\ X - 2Z = 1\,. \end{cases}$$

Solving this system, we obtain X = 1,5; Y = −0,75; Z = 0,25.

Answer: $\bar{x} = 1{,}5\,\bar{p} - 0{,}75\,\bar{q} + 0{,}25\,\bar{r}$.

Example 2

Let A(1; 1; 1), B(5; 1; –2), C(7; 9; 1) be the vertices of the triangle ABC, D be the intersection point of angle A bisectrix with the side BC. Find coordinates of the point D.

Solution. Consider the vectors

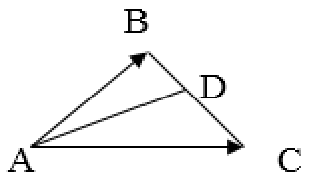
$$\overline{AB} = (4; 0; -3), \qquad \overline{AC} = (6; 8; 0).$$

The magnitudes of these vectors are

$$|\overline{AB}| = \sqrt{4^2 + 0^2 + (-3)^2} = 5;$$

$$|\overline{AC}| = \sqrt{6^2 + 8^2 + 0^2} = 10.$$

By the property of the bisectrix of the triangle we have:



$$\lambda = \frac{|\overline{BD}|}{|\overline{DC}|} = \frac{|\overline{AB}|}{|\overline{AC}|} = \frac{1}{2}$$

then

$$x_D = \frac{x_B + \lambda x_C}{1 + \lambda} = \frac{5 + \frac{1}{2} \cdot 7}{\frac{3}{2}} = \frac{17}{3};$$

$$y_D = \frac{y_B + \lambda y_C}{1 + \lambda} = \frac{1 + \frac{1}{2} \cdot 9}{\frac{3}{2}} = \frac{11}{3};$$

$$z_D = \frac{z_B + \lambda z_C}{1 + \lambda} = \frac{-2 + \frac{1}{2} \cdot 1}{\frac{3}{2}} = -1.$$

Answer: $D\left(\frac{17}{3}; \frac{11}{3}; -1\right)$.

Example 3

The vector $\bar{x}$ is orthogonal to the vectors $\bar{a} = (3; -2; 1)$ and $\bar{b} = (1; 2; -1)$. Find this vector, if $(\bar{x}, \bar{c}) = 12$ where $\bar{c} = (3; 5; -2)$.

Solution. Let the vector $\bar{x}$ have coordinates $\bar{x} = (X; Y; Z)$. Since $\bar{x}$ is orthogonal to $\bar{a}$ and by the orthogonality property of vector $\bar{x}$ to vectors $\bar{a}$ and $\bar{b}$ we have

$$3X - 2Y + Z = 0,$$

$$X + 2Y - Z = 0.$$

The equality $(\bar{x}, \bar{c}) = 12$ means

$$3X + 5Y - 2Z = 12.$$

Solving the system

$$\begin{cases} 3X - 2Y + Z = 0; \\ X + 2Y - Z = 0; \\ 3X + 5Y - 2Z = 12 \end{cases}$$

we obtain X=0, Y=12, Z=24.

Answer: $\bar{x} = (0; 12; 24)$.

Example 4

Let's calculate the area of the parallelogram with diagonal vectors $\bar{p} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$ and $\bar{q} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$.

Solution. We calculate the area of the parallelogram by the formula

$$S = \frac{1}{2}|\bar{p}||\bar{q}| \cdot \sin\alpha,$$

where α is the angle between the diagonals of parallelogram.

Note that

$$|\bar{p}||\bar{q}| \cdot \sin\alpha = |[\bar{p}, \bar{q}]| = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

where $\bar{p} = (x_1, y_1, z_1), \bar{q} = (x_2, y_2, z_2)$.

We have for the area of the parallelogram:

$$S = \frac{1}{2} mod \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \frac{1}{2}\left|3\bar{\imath} - 3\bar{\jmath} - 3\bar{k}\right| = \frac{1}{2}\sqrt{3^2 + (-3)^2 + (-3)^2} = \frac{3\sqrt{3}}{2}.$$

Answer: $S = \frac{3\sqrt{3}}{2}$.

Example 5

Find the altitude of the triangular pyramid with apex O(0;0;0) and vertices A(2;-1;1), B(5;5;4), C(3;2;-1) are the vertices of this pyramid.

Solution. Since the volume of the triangular pyramid is calculated by the formula $V = \frac{1}{3} S_{ABC.} H,$ we get

$$H = \frac{3V}{S_{ABC.}}.$$

Let's calculate the coordinates of vectors with A as the start point:

$$\overline{AO} = (-2; 1; -1), \qquad \overline{AB} = (3; 6; 3), \qquad \overline{AC} = (1; 3; -2).$$

The volume of the triangular pyramid is calculated by the formula

$$V = \frac{1}{6} |\overline{AO} \cdot \overline{AB} \cdot \overline{AC}|.$$

We have

$$\overline{AO} \cdot \overline{AB} \cdot \overline{AC} = \begin{vmatrix} -2 & 1 & -1 \\ 3 & 6 & 3 \\ 1 & 3 & -2 \end{vmatrix} = 48.$$

Then $V = \frac{1}{6} \cdot 48 = 8.$

We know that $S_{ABC} = \frac{1}{2} |[\overline{AB}; \overline{AC}]|.$ Then

$$[\overline{AB}; \overline{AC}] = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 3 & 6 & 3 \\ 1 & 3 & -2 \end{vmatrix} = -21\bar{\imath} + 9\bar{\jmath} + 3\bar{k};$$

$$S_{ABC} = \frac{1}{2}\sqrt{21^2 + 9^2 + 3^2} = \frac{\sqrt{531}}{2} = \frac{3\sqrt{59}}{2}.$$

So

$$H = \frac{3V}{S_{\text{осн.}}} = \frac{3 \cdot 8 \cdot 2}{3\sqrt{59}} = \frac{16}{\sqrt{59}}.$$

Answer: $H = \frac{16}{\sqrt{59}}.$

3. HOME TASKS

I. Solve the following tasks.

1. Show that three vectors $\bar{e}_1 = (1;\ 0;\ 0)$, $\bar{e}_2 = (1;\ 1;\ 0)$, $\bar{e}_3 = (1;\ 1;\ 1)$ form a basis. Calculate the coordinates of the vector $\bar{a} = \bar{j} - \bar{k}$ in the basis of $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
2. Coordinates of the vectors $\bar{a} = (2;\ 1;\ 0)$, $\bar{b} = (1;\ –1;\ 2)$, $\bar{c} = (2;\ 2;\ –1)$ are given. Decompose the vector $\bar{d} = (1;\ 2;\ –1)$ by the vectors $\bar{a}, \bar{b}, \bar{c}$.
3. Decompose the vector $\bar{x} = \bar{a} - \bar{c}$ by three vectors: $\bar{p} = \bar{a} + \bar{b}$, $\bar{q} = \bar{b} + \bar{c}$, $\bar{r} = \bar{c} + \bar{a}$, where $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors.
4. Decompose the vector $\bar{x} = \bar{a} + \bar{b}$ for three vectors: $\bar{b} + \bar{c} - \bar{a}$, $\bar{a} - \bar{b} + \bar{c}$, $\bar{a} + \bar{b} - \bar{c}$, where $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors.
5. Show that three vectors $\bar{e}_1 = (1;\ 0;\ 1)$, $\bar{e}_2 = (0;\ 1;\ 1)$, $\bar{e}_3 = (1;\ 1;\ 0)$ forms a basis. Calculate the coordinates of the vector $\bar{a} = \bar{\imath} + \bar{j} + 2\,\bar{k}$ in the basis of $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
6. Decompose the vector $\bar{d} = (0;\ 1;\ 2)$ by the vectors of basis $\bar{a}, \bar{b}, \bar{c}$, where $\bar{a} = (1;\ 0;\ -2), \bar{b} = (0;\ 2;\ 3), \bar{c} = (1;\ 1;\ 1)$.
7. Given coordinates of 4 points A(1; –2), B(2; 1), C(3; 2) and D(–2; 3). Decompose the vector $\overline{AD}$ by the vectors of basis $\overline{AB} = \bar{a}$, $\overline{AC} = \bar{b}$.
8. Decompose vector $\bar{x} = \bar{a} + \bar{b} - \bar{c}$ by the vectors $\bar{p} = \bar{a} + \bar{b}$, $\bar{q} = \bar{a} + \bar{b} + \bar{c}$, $\bar{r} = \bar{b} + \bar{c}$, where $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors.
9. Decompose the vector $\bar{d} = 2\bar{\imath} + 3\bar{j} - \bar{k}$ by the vectors $\bar{a} = \bar{\imath} + \bar{j} + \bar{k}$, $\bar{b} = 2\bar{\imath} - \bar{j} + \bar{k}$, $\bar{c} = 2\bar{\imath} - 2\bar{j}$.
10. Show that three vectors $\bar{e}_1 = (1;\ 1;\ 1)$, $\bar{e}_2 = (1;\ 1;\ 0)$, $\bar{e}_3 = (0;\ 1;\ 1)$ forms a basis. Decompose the vector $\bar{x} = (1; 2; 3)$ by the basis of $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
11. Given coordinates of 5 points A(1; 0; 1), B(3; 2; 0), C(1; –1; –1), D(0; 1; 0), O(1; 0; 0). Decompose the vector $\overline{BD}$ in the basis of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$.
12. Decompose the vector $\bar{d} = (0;\ 2;\ 1)$ by the vectors of basis $\bar{a} = (-1;\ 0;\ 1)$, $\bar{b} = (1;\ 1;\ 0), \bar{c} = (1;\ 1;\ 1)$.

13. Show that three vectors $\bar{e}_1 = (1;\ 0;\ 0)$, $\bar{e}_2 = (1;\ 1;\ 0)$, $\bar{e}_3 = (0;\ 1;\ 1)$ forms a basis. Calculate the coordinates of vector $\bar{a} = 2\bar{\imath} + \bar{\jmath}$ in the basis of $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
14. Decompose the vector $\bar{x} = \bar{a} - \bar{b} + \bar{c}$ by three vectors: $\bar{p} = \bar{a} + \bar{b} - 2\bar{c}$, $\bar{q} = \bar{a} - \bar{b}$, $\bar{r} = 2\bar{b} + 3\bar{c}$, where $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors.
15. Decompose the vector $\overline{BD}$ by the vectors of basis $\overline{AB} = \bar{a}, \overline{AC} = \bar{b}$, where $A(1;-2), B(\ 2;\ 1), C(3;\ 2), D(-2;3)$.
16. Show that three vectors $\bar{e}_1 = (1;\ 0;\ 1)$, $\bar{e}_2 = (0;\ 1;\ 1)$, $\bar{e}_3 = (1;\ 1;\ 1)$ forms a basis. Calculate the coordinates of the vector $\bar{a} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$ in the basis of $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
17. Decompose the vector $\bar{p} = \bar{a} - \bar{c}$ by the vectors of basis $\bar{a}, \bar{b}$, where $\bar{a} = (3;\ 1), \bar{b} = (0;\ 2), \bar{c} = (4;\ 7)$.
18. Decompose the vector $\overline{BC}$ by the vectors $\overline{AB}, \overline{CD}$, where $A(1;0), B(\ 2;\ 3), C(-1;\ 1), D(3;-1)$.
19. Decompose the vector $\overline{FA}$ by the vectors $\overline{FB}$, $\overline{FC}$, $\overline{FD}$ where $A(1;2;1), B(\ 1;0;2), C(3;\ 2;0), D(-1;-2;3). F(0;4;0)$.
20. Decompose the vector $\bar{p} = \bar{a} + \bar{b} + \bar{c}$ by the vectors of basis $\bar{a}$ and $\bar{b} - \bar{c}$, where $\bar{a} = (1;3)$, $\bar{b} = (1;1)$, $\bar{c} = (2;1)$.
21. Let $\bar{c} = 4\bar{\imath} - 3\bar{k}$ be the decomposition of the vector $\bar{c}$ by basis $\bar{\imath}, \bar{\jmath}, \bar{k}$. Let $\bar{c}$ and $\bar{d}$ be parallel vectors. Find decomposition by this basis for the vector $\bar{d}$, if $|\bar{d}| = 25$.
22. Vectors $\bar{a} = (1;\ 2)$, $\bar{b} = (3;\ 1)$, $\bar{c} = (-1;\ 4)$ are given. Decompose the vector $\bar{p} = 2\bar{a} + \bar{b} - \bar{c}$ by the vectors of basis $\bar{a}$ and $\bar{b}$.
23. Decompose the vector $\overline{CD}$ by the vectors $\overline{AB} = \bar{a}$, $\overline{AC} = \bar{b}$, where $A(1;-2), B(\ 2;\ 1), C(3;\ 2), D(-2;3)$.
24. Let $\bar{a} = 2\bar{\imath} - 2\bar{\jmath} + 2\sqrt{2}\,\bar{k}$ be the decomposition of vector $\bar{c}$ by basis $\bar{\imath}, \bar{\jmath}, \bar{k}$. Let $\bar{a}$ and $\bar{b}$ are anti-parallel vectors. Find decomposition by this basis for the vector $\bar{b}$, if $|\bar{b}| = 24$.
25. Decompose the vector $\overline{AD} + \overline{BD} + \overline{CD}$ by the vectors of basis $\overline{AB} = \bar{a}$, $\overline{AC} = \bar{b}$, where $A(1;-2), B(\ 2;\ 1), C(3;\ 2), D(-2;3)$.

II. Solve tasks.

1. Let A(2; –4; 5), B(–3; 2; 7), point C belong to the axis OX and AC=BC. Find the coordinates of the point C.
2. Let A(3; 3; 3), B(–1; 5; 7), the points C and D divide the segment AB into three equal pieces. Find coordinates of the points C and D.
3. Let A(1; 2; 3), B(7; 10; 3), C(–1; 3; 1) be the vertices of the triangle. Show that A is an obtuse angle.
4. Let A(2; 4; 1), B(–3; 2; 5), point C belong to the axis OZ and AC=BC. Find coordinates of the point C.
5. Let A(1; –1; 5), B(3; 4; 4), C(4;6;1), point D belong to the plane XOY and AD=BD=CD. Find the coordinates of the point D.
6. Let A, B, C be the vertices of a triangle and $\overline{r_A} = \bar{\imath} + 2\bar{\jmath} + 3\bar{k}$, $\overline{r_B} = 3\bar{\imath} + 2\bar{\jmath} + \bar{k}$, $\overline{r_C} = \bar{\imath} + 4\bar{\jmath} + \bar{k}$ be their radius-vectors. Prove that ABC is an equilateral triangle.
7. Let $\bar{a} = \bar{\imath} + 2\bar{\jmath} + \bar{k} - \frac{1}{5}\left(4\bar{\imath} + 8\bar{\jmath} + 3\bar{k}\right)$. Find the magnitude and the direction cosines of the vector $\bar{a}$.
8. Let M_1(1; 2; 3) and M_2(3; –4; 6). Find the magnitude and the direction of the vector $\overline{M_1M_2}$.
9. Let P(3; 5) and Q(1; –3) be two opposite vertices of the square. Find its area.
10. Let A(–3; 2) and B(1; 6) be two vertices of equilateral triangle. Find its area.
11. Let A(1; 4), B(3; –9), C(–5; 2) be the vertices of triangle. Find the length of the median drawn from the vertex B.
12. Prove that the points A(3; –5), B(–2; –7) and C(18; 1) belong to a single line.
13. Let N(2; –3), point M belong to the axis OX and $MN = 5$. Find coordinates of the point M.
14. Let N(–8, 13), point M belong to the axis OY and $MN = 17$. Find coordinates of the point M.

15. Let A(3; –5), B(5; –3), C(–1; 3) be three vertices of the parallelogram. Find fourth vertex D that is opposite to B.
16. Let A(–3; 5), B(1; 7) be two adjacent vertices of the parallelogram, M(1; 1) is the point of intersection of its diagonals. Find two another vertices of this parallelogram.
17. Let A(2; 3), B(4; –1), C(0; 5) be three vertices of the parallelogram.ABCD. Find the fourth vertex D.
18. Let points P(2; 2) and Q(1; 5) divide the segment AB into three equal pieces. Find coordinates of the points A and B.
19. Let $\bar{a} = (3;\ 4;\ -12)$. Find its magnitude, direction and orth.
20. The vector $\bar{a}$ forms the angles α = 60^0, β = 120^0 with the axes OX and OY, $|\bar{a}| = 2$. Find its coordinates in the space OXYZ.
21. Let M(1; 2; –3) be the start point and N be the end point of the vector $\bar{a} = (3;\ -1;\ 4)$. Find coordinates of the point N.
22. Let A(0; 0; 0), B(8; 3; 0), C(–2; 5; 1) be the vertices of a triangle. Find vectors of its medians.
23. Let A(2; 3; 4), B(3; 1; 2), C(4; –1; 3) be the vertices of a triangle. Find the coordinates of triangle's center of gravity.
24. The side of rhombus is equal to $5\sqrt{10}$ and the points P(4; 9), Q(–2; 1) are its opposite vertices. Find the area of this rhombus.
25. Let A(2; –5), B(1; –2), C(4; 7) be the vertices of the triangle. Find the intersection point of the bisectrix of the inner angle B with the side AC.

III. Solve tasks.

1. What condition must the vectors $\bar{a}$ and $\bar{b}$ satisfy, if the vectors $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ are collinear?
2. Let $|\bar{a}| = 3, |\bar{b}| = 4$ and the angle between $\bar{a}$ and $\bar{b}$ be equal to $\frac{2\pi}{3}$. Calculate $(3\bar{a} - 2\bar{b}, \bar{a} + 2\bar{b})$.

3. Let $\bar{a} = (3;\ -1;\ -2)$ and $\bar{b}\ =\ (1;\ 2;\ -1)$. Find coordinates of the vector product $[2\bar{a} - \bar{b}, 2\bar{a} + \bar{b}]$.
4. Find $\text{proj}_{\bar{c}}(2\bar{a}\ - 3\bar{b})$, if $\bar{a} = (-1;\ 2;\ 1)$, $\bar{b}\ = (3;\ 1;\ 1)$, $\bar{c}\ = (4;\ 3;\ 0)$.
5. The vector $\bar{x}$ is collinear to the vector $\bar{a} = (2; 1; -1)$ and $|\,\bar{x}\,| = 2\sqrt{6}$. Find this vector, if it forms obtuse angle with the axis OY.
6. Let $|\,\bar{a}\,|\ =\ 1, |\,\bar{b}\,|\ =\ 2$ and the angle between $\bar{a}$ and $\bar{b}$ be equal to $\frac{2\pi}{3}$. Calculate $\left|[\bar{a}\ + 3\bar{b}, 3\bar{a} - \bar{b}]\right|$.
7. The vector $\bar{x}$ is collinear to the vector $\bar{a} = (2; 1; -1)$ and $(\bar{x}, \bar{a}) = 3$. Find $\bar{x}$.
8. Find $\text{proj}_{\bar{c}}(\bar{a}\ + \bar{b})$, if $\bar{a} = (3;\ -6;\ 1)$, $\bar{b}\ = (1;\ 4;\ -5)$, $\bar{c}\ = (3;\ -4;\ 12)$.
9. What condition must the vectors $\bar{a}$ and $\bar{b}$ satisfy, if vectors $\bar{a}\ + \bar{b}$ and $\bar{a} - \bar{b}$ are perpendicular?
10. The vector $\bar{x}$ is perpendicular to the vectors $\bar{a} = (4; -2; -3)$ and $\bar{b} = (0; 1; 3)$ and it forms obtuse angle with the axis OY. Find its coordinates, if $|\bar{x}\,| = 26$.
11. The vectors $\bar{a}$ and $\bar{b}$ are orthogonal and $|\,\bar{a}\,|\ =\ 3, |\,\bar{b}\,|\ =\ 4$. Calculate $\left|[\bar{a}\ + \bar{b}, \bar{a} - \bar{b}]\right|$.
12. Let $|\,\bar{a}\,|\ =\ 2, |\,\bar{b}\,|\ =\ \sqrt{2}$ and the angle between $\bar{a}$ and $\bar{b}$ be equal to $\frac{3\pi}{4}$. Calculate $(\bar{a} + 3\bar{b}, 2\bar{a} - \bar{b})$.
13. Find $\text{proj}_{2\bar{a}\,-3\bar{b}}\bar{c}$, if $\bar{a} = (-1;\ 3;\ 2)$, $\bar{b}\ = (3;\ 4;\ 1)$, $\bar{c}\ = (3;\ 0;\ 1)$.
14. What are the values α and β if vectors $\bar{a} = (-2; 3; \beta)$ and $\bar{b} = (\alpha; -6; 2)$ are collinear?
15. The vector $\bar{x}$ is collinear to the vector $\bar{a} = (3; -5; 4)$ and $|\,\bar{x}\,|\ =\ 15\sqrt{2}$. Find $\bar{x}$, if it forms obtuse angle with the axis OZ.
16. The vector $\bar{x}$ is orthogonal to vectors $\bar{a} = (2; 1; -1)$ and $\bar{b} = (3; 2; 1)$. Find $\bar{x}$ if $(\bar{x}, \bar{c}) = 17$, where $\bar{c} = (3; -4; 5)$.
17. Find $\text{proj}_{\bar{b}\,+\bar{c}}\bar{a}$, if $\bar{a} = (1;\ -2;\ 3)$, $\bar{b}\ = (1;\ -5;\ 4)$, $\bar{c}\ = (1;\ -3;\ 9)$.

18. Let $|\bar{a}| = 4, |\bar{b}| = 6$ and the angle between $\bar{a}$ and $\bar{b}$ be equal to $\frac{\pi}{3}$. Calculate $\left(3\bar{a} - 2\bar{b}, 5\bar{a} - 6\bar{b}\right)$.
19. Find the value λ such that vectors $\bar{a} = 4\bar{\imath} + \lambda\bar{\jmath} + 5\bar{k}$ and $\bar{b} = \lambda\bar{\imath} + 2\bar{\jmath} - 6\bar{k}$ are orthogonal?
20. Let $|\bar{a}| = 1, \left|\bar{b}\right| = 2, |\bar{c}| = 3$ and the angles between $\bar{a}$ and $\bar{b}$, $\bar{a}$ and $\bar{c}$, $\bar{b}$ and $\bar{c}$, be equal to $\frac{\pi}{3}$. Calculate $\left(2\bar{a} + 3\bar{b} + 4\bar{c}, 5\bar{a} + 6\bar{b} + 7\bar{c}\right)$.
21. The unit vector $\bar{x}$ is orthogonal to the vectors $\bar{a} = \bar{\imath} + \bar{\jmath} + 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$. Find $\bar{x}$.
22. The vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ have equal lengths and form pairwise equal angles. Find $\bar{c}$ if $\bar{a} = \bar{\imath} + \bar{\jmath}$, $\bar{b} = \bar{\jmath} + \bar{k}$.
23. Find $\text{proj}_{\bar{a}}\bar{b}$ and $\text{proj}_{\bar{b}}\bar{a}$, if $\bar{a} = 2\bar{\imath} + 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 6\bar{\imath} + 3\bar{\jmath} + 2\bar{k}$.
24. The vectors $\bar{a}$ and $\bar{b}$ are orthogonal and $|\bar{a}| = 3, |\bar{b}| = 4$. Calculate $\left|\left[3\bar{a} - \bar{b}, \bar{a} - 2\bar{b}\right]\right|$.
25. Let $\bar{a} = (3; -1; -2)$ and $\bar{b} = (1; 2; -1)$. Find coordinates of the vector product $[2\bar{a} + \bar{b}, \bar{b}]$.

IV. Solve tasks.

1. Let $\bar{a} = \bar{p} - 3\bar{q}$, $\bar{b} = 5\bar{p} + 2\bar{q}$, $|\bar{p}| = 2\sqrt{2}$, $|\bar{q}| = 3$ and the angle between $\bar{p}$ and $\bar{q}$ be equal to $\frac{\pi}{4}$. Parallelogram is constructed on vectors $\bar{a}$ and $\bar{b}$. Calculate the length of its diagonals.
2. Let A(1; 2; 0), B(3; 0; –3), C(5; 2; 6). Find the area of triangle ABC.
3. Let $\bar{a} = 3\bar{m} - \bar{n}$, $\bar{b} = 2\bar{m} + 3\bar{n}$, $|\bar{m}| = 2$, $|\bar{n}| = \sqrt{3}$ and the angle between $\bar{m}$ and $\bar{n}$ be equal to $\frac{5\pi}{6}$. Parallelogram is constructed on the vectors $\bar{a}$ and $\bar{b}$. Calculate the length of its diagonals.
4. Let A(3; 1; –5), B(1; 2; 4), C(–1; –2; 1) be the vertices of triangle. Find its area.

5. Let A(1; 3; 5), B(–1; 2; 4), C(2; –3; 3) be the vertices of triangle. Find the height drawn from the vertex A to the side BC.
6. Let A(2; 2; 2), B(4; 0; 3), C(0; 1; 0). Find the area of triangle ABC.
7. Let A(1; –1; 2), B(5; –6; 2), C(1; 3; –1) be the vertices of triangle. Find the height drawn from the vertex B to the side AC.
8. Let A_1(1; 1), A_2(2; 3), A_3(5; –1) be the vertices of triangle. Prove that this triangle is right.
9. Prove that the points A(2, 2), B(–1, 6), C(–5, 3), D(–2, –1) are the vertices of the square.
10. Let $\bar{a} = 2\bar{m} + 3\bar{n}$, $\bar{b} = 3\bar{n} - \bar{m}$, $|\overline{m}| = 1$, $|\bar{n}| = 8$ and the angle between $\bar{m}$ and $\bar{n}$ be equal to $\frac{\pi}{6}$. Parallelogram is constructed on the vectors $\bar{a}$ and $\bar{b}$. Calculate its area.
11. Let A(1; 1; –4), B(–5; 3; –5), C(–3; 1; 2), D(4; 0; 1) be the vertices of quadrilateral. Prove that its diagonals are mutually orthogonal.
12. Let $\bar{a} = 6\bar{m} - 3\bar{n}$, $\bar{b} = 3\bar{m} + 2\bar{n}$, $|\overline{m}| = 3$, $|\bar{n}| = 5$ and the angle between $\bar{m}$ and $\bar{n}$ be equal to $\frac{\pi}{4}$. Parallelogram is constructed on the vectors $\bar{a}$ and $\bar{b}$. Calculate its area.
13. The parallelogram is constructed on the vectors $\bar{a} = (2; 1; -1)$ and $\bar{b} = (1; -3; 1)$. Calculate its area and sinus of angle between $\bar{a}$ and $\overline{b}$.
14. Find a dihedral angle between the vectors' planes: $\bar{a} = (1;\ -3;\ 1), \bar{b} = (0;\ 1;\ 2)$ and $\bar{c} = (2;\ -1;\ 3), \bar{d} = (1;\ 0;\ 1)$.
15. Let A(–3; –2; 6), O(0; 0; 0), B(–2; 4; 4) be the vertices of triangle. Find its area and the length of height drawn from the vertex A.
16. Find the angle between vector $\bar{a} = (1;\ -2; 5)$ and the plane of vectors $\bar{b} = (2;\ -1; 3)$, $\bar{c} = (1; 0; 1)$.
17. The parallelogram is constructed on the vectors $\bar{a} = (5; -4; 7)$ and $\bar{b} = (1;\ 1; -2)$. Calculate its area and the length of its diagonals.

18. Let $\overline{AB} = (-1;\ 2;\ -2)$, $\overline{BC} = (4;\ -1;\ 4)$ be the sides of triangle. Calculate its area and the height $\overline{AD}$.
19. Let $\overline{AB} = 2\bar{e}_1 - \bar{e}_2, \overline{BC} = \bar{e}_1 + \bar{e}_2$ be the sides of triangle ABC. Calculate its height $\overline{CH}$, if $\bar{e}_1$ and $\bar{e}_2$ are mutually orthogonal orths.
20. Let $\bar{e}_1$ and $\bar{e}_2$ be unit vectors and the angle between them be equal to $\frac{\pi}{3}$. Let the vectors $\bar{e}_1 + \bar{e}_2$ and $2\bar{e}_1 - \bar{e}_2$ be diagonals of parallelogram. Calculate its area.
21. Let A(3; –4; 7), B(–5; 3; –2), C(1; 2; –3) be three vertices of the parallelogram ABCD. Find its area.
22. Let A(–2; 2), B(1; 4) be two adjacent vertices of the parallelogram, M(2; 0) be the point of intersection of its diagonals. Find the area of the parallelogram.
23. Let $A(-1; 2; 4), B(3;\ -1; 2)$ and $C(5; 1; 3)$ be the vertices of triangle. Prove that this triangle is right.
24. Let $A(1; 2; 3), B(-1;\ 0; 1)$ be two vertices of triangle ABC and the point $M(-1; 2;\ -1)$ be the center of the side AC. Find the area of triangle ABC.
25. Let the vectors $\bar{p} = 2\bar{\imath} + \bar{\jmath} - \bar{k}$ and $\bar{q} = \bar{\imath} + \bar{\jmath} + \bar{k}$ be diagonals of the parallelogram. Find its area.

V. Solve tasks.

1. Are the vectors $\bar{a} = (23; 6; 8), \bar{b} = (3; -4; 2)$, $\bar{c} = (4; 3; 1)$ coplanar?
2. Let $A(1;\ 3;\ -2), B(4; -1; 5), C(3;\ -2; 1), D(-4;\ -1; 3)$ be the vertices of the triangular pyramid. Calculate its volume.
3. Find the scalar triple product of vectors $\bar{a} = \bar{\imath} - \bar{\jmath} + \bar{k}$, $\bar{b} = \bar{\imath} + \bar{\jmath} + \bar{k}, \bar{c} = 2\bar{\imath} + 3\bar{\jmath} + 4\bar{k}$.
4. Are the vectors $\bar{a} = 7\bar{\imath} - 3\bar{\jmath} + 2\bar{k}$, $\bar{b} = 3\bar{\imath} - 7\bar{\jmath} + 8\bar{k}, \bar{c} = \bar{\imath} - \bar{\jmath} + \bar{k}$ coplanar?
5. Let $A(0;\ 0;\ 1), B(2; 3; 5), C(6;\ 2; 3), D(3;\ 7; 2)$ be the vertices of the triangular pyramid. Calculate its height omitted on the face BCD.
6. Prove that the points A (5; 7; – 2), B (3; 1; – 1), C (9; 4; –4) and D (1; 5; 0) belong to a single plane.

7. Let the vectors $\bar{a}, \bar{b}, \bar{c}$ be mutually orthogonal and form the right-handed triple. Calculate $\bar{a}\bar{b}\bar{c}$, if $|\bar{a}| = 4$, $|\bar{b}| = 2$, $|\bar{c}| = 3$.
8. Are the vectors $\bar{a} = (2; 3; -1), \bar{b} = (1; -1; 3)$, $\bar{c} = (1; 9; -11)$ coplanar?
9. Prove that the points A (1; 2; – 1), B (0; 1; 5), C (-1; 2; 1) and D (2; 1; 3) belong to a single plane.
10. Let $A(2;\ -1;\ 1), B(5; 5; 4), C(3;\ 2; -1), D(4;\ 1; 3)$ be the vertices of triangular pyramid. Calculate its volume.
11. Find the scalar triple product of the vectors $\bar{x} = (1; 2; 3)$, $\bar{y} = (0; 1; 1), \bar{z} = [\bar{x}, \bar{y}]$.
12. Find the height of the triangular pyramid omitted from the vertex D,

    if the points A(2; 3; 1), B(4; 1; −2), C(6; 3; 7), D(−5;−4; 8) are the vertices of this pyramid.
13. Are the vectors $\bar{a} = (3; -2; 1), \bar{b} = (2; 1; 2)$, $\bar{c} = (3; -1; -2)$ coplanar?
14. Let A(2;1;−1), B(3; 0; 1), C(2;−1; 3) be three vertices of. the triangular pyramid ABCD. Its volume V = 5. Find the fourth vertex D, if it belongs to the axis OY.
15. What is the value α if the vectors $\bar{a} = (\alpha; 0; 1)$, $\bar{b} = (1; 2\alpha; 0)$, $\bar{c} = (1; 0; \alpha)$ are coplanar?
16. Let $O(1;\ 1; 2), A(2;\ 3;\ -1), B(2; -2; 4), C(-1;\ 1; 3)$ be the vertices of the triangular pyramid. Calculate its volume.
17. Are the vectors $\bar{a} = (2; -1; 2), \bar{b} = (1; 2; -3)$, $\bar{c} = (3; -4; 7)$ coplanar?
18. Find the height of the triangular pyramid omitted from the vertex C,

    if the points A(0; –2; 5), B(6; 6; 0), C(3; –3; 6), D(2;−1; 3) are the vertices of this pyramid.
19. Find the mixed product of the vectors $\bar{a} = (1;\ -1; 3)$, $\bar{b} = (-2; 2; 1)$,

    $\bar{c} = (3; -2; 5)$.
20. Prove that the points A (3; 5; 1), B (2; 4; 7), C (1; 5; 3) and D (4; 4; 5) belong to a single plane.
21. Let $A(1;\ 2;\ 3), B(-2; 4; 1), C(7;\ 6; 3), D(4;\ -3; -1)$ be the vertices of the triangular pyramid. Calculate its volume and the height omitted on the face ABC.

22. Calculate the volume of the triangular pyramid OABC, if $\overline{OA} = \bar{\imath} + 2\bar{\jmath}$, $\overline{OB} = \bar{\imath} + \bar{\jmath} + \bar{k}$, $\overline{OC} = \bar{\jmath} + \bar{k}$.
23. What is the value α if the vectors $\bar{x} = \alpha\bar{\imath} + \bar{\jmath} + \bar{k}$, $\bar{y} = \bar{\imath} + \bar{\jmath} - 2\bar{k}$, $\bar{z} = \bar{\imath} + \alpha\bar{\jmath} + \bar{k}$ are coplanar?
24. Find the volume of the parallelepiped constructed on the vectors $\bar{a} = \bar{\imath} + \bar{\jmath} + \bar{k}$, $\bar{b} = -\bar{\imath} - \bar{\jmath} + \bar{k}$, $\bar{c} = [\bar{a}, \bar{b} + 2\bar{a}]$.
25. Find the scalar triple product of the vectors $\bar{m} = \bar{\imath} + \bar{k}$, $\bar{n} = \bar{\imath} + \bar{\jmath}$, $\bar{l} = [2\bar{m} + \bar{n}, \bar{n} - \bar{m}]$.

VI. Solve tasks.

1. Prove the identity: $[\bar{a} + \bar{b}, \bar{a} - \bar{b}] = 2[\bar{b}, \bar{a}]$.
2. Simplify the expression: $(3\bar{\imath}, [\bar{\jmath}, \bar{k}]) + 5(\bar{\jmath}, [\bar{\imath}, \bar{k}]) + 2(\bar{k}, [\bar{\imath}, \bar{\jmath}])$.
3. Simplify the expression: $[5\bar{a} - \bar{b}, \bar{c} + \bar{a}] + [\bar{b} + 5\bar{c}, \bar{a} + \bar{b}]$.
4. Prove the identity: $(2\bar{a} + \bar{b})(\bar{b} - \bar{c})(\bar{c} + \bar{a}) = \bar{a}\bar{b}\bar{c}$.
5. Simplify the expression: $[\bar{\imath} + \bar{\jmath}, 10\bar{\imath} - \bar{k}] + [2\bar{\imath} + \bar{\jmath}, 5\bar{\jmath} + \bar{k}]$.
6. Prove the identity: $\bar{a}\bar{b}(\bar{c} - 10\bar{b} + \bar{a}) = \bar{a}\bar{b}\bar{c}$.
7. Simplify the expression: $[\bar{\imath} + \bar{\jmath}, \bar{k} - \bar{\imath}] - [\bar{\jmath}, \bar{k} + \bar{\imath}] + [\bar{k}, \bar{\imath} + \bar{\jmath} + 2\bar{k}]$.
8. Prove the identity: $(2\bar{a} + \bar{b} + \bar{c})(\bar{b} - 2\bar{c})(\bar{a} + 2\bar{c}) = \bar{a}\bar{b}\bar{c}$.
9. Simplify the expression: $[2\bar{k} - 5\bar{\imath}, \bar{\jmath} + 3\bar{\imath}] + [\bar{\imath} - \bar{\jmath}, 5\bar{\jmath} + 6\bar{k}]$.
10. Prove the identity: $[\bar{a} - \bar{b} + \bar{c}, \bar{a}] - [\bar{b}, \bar{a} + \bar{c}] + [\bar{a} + \bar{b} - \bar{c}, \bar{c}] = 2[\bar{a}, \bar{b}]$.
11. Simplify the expression: $(\bar{a} - \bar{b})(\bar{b} - \bar{c})(\bar{c} - \bar{a})$.
12. Prove the identity: $[\bar{\imath}, \bar{\jmath} + \bar{k} - \bar{\imath}] + [\bar{\jmath} + 2\bar{k}, 2\bar{k} - \bar{\imath}] + [\bar{\imath} + \bar{\jmath} - \bar{k}, \bar{\jmath}] = 3(\bar{\imath} - \bar{\jmath} + \bar{k})$.
13. Simplify the expression: $([\bar{\imath}, \bar{\jmath}] + \bar{k}, [\bar{k} + \bar{\imath}, \bar{\jmath}] - [\bar{\jmath}, 2\bar{k} - \bar{\imath}])$.
14. Prove the identity: $(\bar{a} + \bar{b} + \bar{c})(\bar{c} + \bar{b} - \bar{a})(\bar{a} + \bar{b} + \bar{c}) = 0$.
15. Simplify the expression: $(\bar{a} + \bar{b})(\bar{b} + \bar{c})(\bar{c} - \bar{a})$.
16. Prove the identity: $(\bar{b} + 2\bar{c})(2\bar{a} + \bar{b})(\bar{a} + \bar{c}) = 0$.
17. Simplify the expression: $(\bar{\imath} + 2\bar{k})(\bar{\jmath} - \bar{\imath} - \bar{k})(3\bar{\imath} + \bar{k})$.

18. Prove the identity: $(\bar{a} + 2\bar{b} - \bar{c})(3\bar{a} - \bar{b} + \bar{c})(5\bar{b} - \bar{a} - 3\bar{c}) = 0.$

19. Simplify the expression: $[\bar{a} + \bar{b}, 10\bar{a} - \bar{c}] + [2\bar{a} + \bar{b}, 5\bar{b} + \bar{c}].$

20. Prove the identity: $[18\bar{a} + \bar{b} - \bar{c}, 2\bar{a}] - [\bar{b} + 4\bar{c}, 2\bar{b} + 2\bar{a}] + [\bar{c}, 10\bar{a} + 8\bar{b}] = \bar{0}.$

21. Simplify the expression: $[\bar{\imath}, \bar{\jmath} - \bar{k}] + [\bar{\jmath} + \bar{\imath}, \bar{\imath} - \bar{k}] - [\bar{k}, \bar{\imath} + \bar{\jmath}].$

22. Simplify the expression: $[10\bar{a} + 6\bar{c} - \bar{b}, \bar{c}] - [\bar{a}, 10\bar{c} + 12\bar{a}].$

23. Simplify the expression: $[12\bar{a} + \bar{b}, 10\bar{a} - \bar{c}] + [4\bar{a} + 5\bar{b}, 3\bar{c} - 2\bar{a}].$

24. Prove the identity: $(\bar{\imath} + \bar{\jmath})(\bar{k} + \bar{\imath})(\bar{\jmath} - \bar{k}) = 0.$

25. Simplify the expression: $(\bar{a} + 2\bar{b})(\bar{b} - \bar{a} - \bar{c})(3\bar{a} + \bar{c}).$

LIST OF RECOMMENDED LITERATURE

1. Konev V.V. Linear algebra, vector algebra and analytical geometry. Textbook. – Tomsk: TPU press, 2009. – 114pp.
2. Kishan H. Vector algebra and calculus. – Atlantic Publishers & Distributors (P) Ltd, 2007. – 440pp.
3. Vector algebra – simpleNeasyBook (Kindle Edition). – WAGmob, 2013. – 439pp.

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